Constrained Inefficiency in a Huggett Model

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1 Introduction

This paper discusses the efficiency properties of equilibria in a Huggett (1993) model. Our analysis is motivated by the literature on the constrained inefficiency of models of general equilibrium with exogenously incomplete markets. Geanakoplos and Polemarchakis (1986) proved the constrained suboptimality of competitive equilibrium allocations in models with uncertainty and an exogenously incomplete asset market. Generically, there is a reallocation of assets that leads to a Pareto superior allocation of goods after prices in commodity spot markets adjust and markets clear. More recently, Davila, Hong, Krusell and Rios-Rull (working paper) show that the competitive equilibrium of an Aiyagari (1994) model is constrained inefficient. In the simplest case with two periods and in which agents are identical in the first period, they show that the planner can improve upon the laissez-faire equilibrium by imposing everyone a lower savings level, which has the effect of lowering wages and raising interest rates, thereby reducing the agents’ exposition to uninsurable risk.

It is interesting to study whether a Huggett economy is constrained inefficient, and if so, whether the equilibrium interest rate is too high or too low. In this economy, as opposed to the Aiyagari’s framework, the planner cannot impose everyone to save more (or less) than in the competitive equilibrium, because the market clearing constraint has to remain satisfied. Thus the question we ask is whether, starting from an equilibrium of the economy with borrowing constraint $a < 0$, a fictitious planner can improve on the allocation by simply imposing a different borrowing constraint $a'$, while respecting all budget constraints of agents and letting markets operate freely under perfect competition. In particular, the planner is not allowed to complete the markets or in any way transfer goods between lucky and unlucky consumers; the only insurance asset is still the riskless bond.

In section 2 of the paper, we study a two-period Huggett model, and prove that the equilibrium is very generally constrained inefficient. We characterize the constrained optimum as the cutoff such that the interest rate is too high if the borrowing constraint is lower than the cutoff (which is the case if we only impose the natural debt limit), and too low otherwise. Our result suggests that the constrained inefficiency property of incomplete markets models is not due to an inefficient level of capital accumulation, as in
Davila et al.; even in a model without aggregate capital, the price of the asset is not set optimally in the laissez-faire economy. Another result we obtain in the two-period model is that the constrained inefficiency disappears when both income shocks are permanent. This means that even though there is no insurance at all in the competitive equilibrium, the planner cannot improve on the allocation. He can do so only if the agents effectively trade on the bond market to self-insure, which will happen if at least one of the income shocks is not fully permanent.

We then move to an infinite period economy in section 3, in order to see whether our main result (the constrained inefficiency of the equilibrium) goes through. We look numerically at whether the planner can improve on the steady-state allocation by tightening the borrowing constraint. We find that the infinite-horizon economy is indeed constrained inefficient.

We conclude and discuss extensions in section 4.

2 A Two-Period Economy

2.1 The Setup

Consider a Huggett economy with two periods, \( t \in \{1, 2\} \). There is a continuum of consumers with measure 1. In every period \( t \), an individual \( i \)'s income \( y_i^t \in \{y_l, y_h\} \), with \( 0 < y_l < y_h \). In period 1, a fraction \( \pi \) of the agents have income \( y_h \) and a fraction \( 1 - \pi \) have income \( y_l \). Income follows an exogenous first-order Markov process with \( \mathbb{P}(y_{t+1} = y_h | y_t = y_h) = \pi_h \) and \( \mathbb{P}(y_{t+1} = y_l | y_t = y_l) = \pi_l \). We assume that \( \pi_h \in [0, 1] \), \( \pi_l \in [0, 1] \), and \( (\pi_h, \pi_l) \neq (1, 1) \), i.e. at least one type of agents faces some uncertainty regarding his income in the second period. Finally, we assume that \( \pi_l \geq 1 - \pi_h \).

The utility function \( u : \mathbb{R}_+ \to \mathbb{R} \) is twice continuously differentiable, strictly increasing and strictly concave in consumption, \( u' > 0 \) and \( u'' < 0 \), and satisfies the Inada condition \( u'(c) \to c \to 0 \infty \). We also assume that \( u' \) is convex, i.e. \( u''' > 0 \).

In every period \( t \), agents can accumulate a risk-free bond \( a_{t+1} \) at price \( q_t \), subject to a borrowing constraint: \( \forall t, a_t \geq a \). At the beginning of period 1, agents have no initial wealth: \( a_1^h = a_1^l = 0 \). The net supply of bonds in the economy is zero.

We can interpret our model as a three-period economy in which all individuals are identical in period 0. Since the asset market has to clear, no one saves or borrows, and all individuals arrive in period \( t = 1 \) with zero assets. A fraction \( \pi \) of them have a positive income shock in period 1, and the others receive a negative shock. This interpretation of our model will be useful when we choose a social welfare function.

Definition 2.1. We define an equilibrium \( E(a) \equiv (a^h, a^l, q) \) for this economy as follows:
1. Agents $h$ choose $a^{h*}$ given $q^*$ and $a$, which solves

$$a^{h*} \in \arg \max_{a^{h} \geq a} u \left( y_h - q^* a^{h*} \right) + \beta \left\{ \pi_h u \left( y_h + a^{h*} \right) + (1 - \pi_h) u \left( y_l + a^{h*} \right) \right\}$$

2. Agents $l$ choose $a^{l*}$ given $q^*$ and $a$, which solves

$$a^{l*} \in \arg \max_{a^{l} \geq a} u \left( y_l - q^* a^{l*} \right) + \beta \left\{ \pi_l u \left( y_l + a^{l*} \right) + (1 - \pi_l) u \left( y_h + a^{l*} \right) \right\}$$

3. Asset market clearing is satisfied:

$$\pi a^{h*} + (1 - \pi) a^{l*} = 0 \quad \text{(2.1)}$$

Let $V^h(a), V^l(a)$ be the value functions of agents $h$ and $l$, respectively.

### 2.2 Equilibrium

The three necessary conditions for an equilibrium are agent $h$’s first-order condition (2.2), agent $l$’s first-order condition (2.3), and market clearing (2.1).

1. Agent $h$’s optimal savings choice $a^{h*}$ satisfies

$$q^* u' \left( y_h - q^* a^{h*} \right) \geq \beta \pi_h u' \left( y_h + a^{h*} \right) + \beta (1 - \pi_h) u' \left( y_l + a^{h*} \right) \quad \text{(2.2)}$$

with equality if $a^{h*} > a$.

2. Agent $l$’s optimal savings choice $a^{l*}$ satisfies

$$q^* u' \left( y_l - q^* a^{l*} \right) \geq \beta \pi_l u' \left( y_l + a^{l*} \right) + \beta (1 - \pi_l) u' \left( y_h + a^{l*} \right) \quad \text{(2.3)}$$

with equality if $a^{l*} > a$.

3. Market clearing holds:

$$\pi a^{h*} + (1 - \pi) a^{l*} = 0$$

**Assumption 1 (Existence).** We assume that for all $a \leq 0$, there exists a solution \{${a^{h*}(a), a^{l*}(a), q^*(a)}$\} to this system of equations.

**Lemma 2.1.** In any equilibrium with $a < 0$, we have $a \leq a^{l*} < 0 < a^{h*}$.

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1 For ease of notations, we omit subscripts from now on and let $q \equiv q_2$ and $a \equiv a_3$. 
Proof. Let \( a < 0 \), and let \((a^h, a^l, q^*)\) be an equilibrium for this economy.

Suppose first that \( a^l = a^{h*} = 0 \), so that market clearing is satisfied. Since \( a < 0 \), this implies that the borrowing constraint binds neither for agent \( h \) nor for agent \( l \). Hence both first-order conditions (2.2) and (2.3) hold with equality. Using (2.3) and the facts that \( y_h > y_l \) and that \( u \) is strictly concave, we have

\[
q^* u'(y_l) = \beta \pi_l u'(y_l) + \beta (1 - \pi_l) u'(y_h) \\
\leq \beta \pi_l u'(y_l) + \beta (1 - \pi_l) u'(y_l) = \beta u'(y_l)
\]

with a strict inequality if \( \pi_l < 1 \). Therefore \( q^* \) has to be smaller than \( \beta \), and strictly smaller if \( \pi_l < 1 \). But the same reasoning applied to (2.2) implies that

\[
q^* u'(y_h) = \beta \pi_h u'(y_h) + \beta (1 - \pi_h) u'(y_l) \\
\geq \beta \pi_h u'(y_h) + \beta (1 - \pi_h) u'(y_h) = \beta u'(y_h)
\]

with a strict inequality if \( \pi_h < 1 \). Therefore \( q^* \) has to be larger than \( \beta \), and strictly larger if \( \pi_h < 1 \). Because we assumed that we have \((\pi_h, \pi_l) \neq (1, 1)\), at least one of these inequalities is strict, and we obtain a contradiction.

Suppose next that \( a^l > 0 > a^{h*} \). Then the borrowing constraint does not bind for agent \( l \), so that (2.3) holds with equality. By a reasoning similar as in the previous case, and using \( a^{h*} > 0 \) and \( q^* > 0 \), we have

\[
q^* u'(y_l - q^* a^l) = \beta \pi_l u'(y_l + a^l) + \beta (1 - \pi_l) u'(y_h + a^{h*}) \\
\leq \beta \pi_l u'(y_l + a^l) + \beta (1 - \pi_l) u'(y_l + a^{l*}) = \beta u'(y_l + a^{l*}) \\
< \beta u'(y_l - q^* a^l)
\]

Thus we must have \( q^* < \beta \). Symmetrically, we have, using the inequality (2.2) and \( a^{h*} < 0 \),

\[
q^* u'(y_h - q^* a^{h*}) \geq \beta \pi_h u'(y_h + a^{h*}) + \beta (1 - \pi_h) u'(y_h + a^{h*}) \\
= \beta u'(y_h + a^{h*}) > \beta u'(y_h - q^* a^{h*})
\]

But this implies \( q^* > \beta \), a contradiction.

Therefore we must have \( a^l < 0 < \beta a^{h*} \). \( \square \)

We call \( a = -y \) the natural borrowing constraint. Note that it is possible that the system of equations (2.1), (2.2), (2.3) has more than one solution, i.e. we do not rule out the possibility of multiple equilibria. We assume that there is a countable number of equilibria for each \( a < 0 \). If for a given \( a < 0 \) there are several equilibria, we index them by \( E_1(a), \ldots, E_n(a) \) with corresponding optimal choices \( a^i_1 < \ldots < a^i_n < 0 \) and \( a^{h*} > \ldots > a^{h*} > 0 \) and prices \( q^i_1, \ldots, q^i_n \). \(^2\) In the next section we study the property of

\(^2\)We will prove in the next section that \( q^*_1 > \ldots > q^*_n \).
constrained inefficiency of these equilibria. We first state a result which characterizes the relationship between the equilibrium price $q^*$ and the discount factor $\beta$. The proof is in the Appendix.

**Lemma 2.2.** There exists $\eta > 0$ such that $\pi_l > 1 - \eta \Rightarrow q^* > \beta$.

*Proof.* See Appendix. \qed

Note that if $\pi_h = 1$, then by (2.2) we have $q^* < \beta$. So the previous lemma does not generalize to any values of the parameters.

### 2.3 Constrained Inefficiency

Following the literature on general equilibrium with incomplete markets, we now discuss the efficiency properties of an equilibrium. We study whether our simple Huggett economy is constrained inefficient, and if so, whether the equilibrium interest rate is too high or too low. The question we ask is whether, starting from an equilibrium $E(a)$ of the economy with borrowing constraint $a < 0$, a fictitious planner can improve on the allocation by simply imposing a different borrowing constraint $a'$, while respecting all budget constraints of agents and letting markets operate freely under perfect competition.

Formally, we call the equilibrium constrained efficient if there is no borrowing constraint $a'$ such that, letting markets operate competitively, the utilitarian sum of consumers’ utility is higher than under the competitive equilibrium. We consider a utilitarian social welfare function for two reasons. First, we will see that a tightening of the borrowing constraint makes some consumers (namely, the borrowers) better off, and some consumers (the savers) worse off; thus it is not possible to obtain a Pareto improvement by altering the borrowing constraint. But as Davila et al. argue, the natural planner’s objective is that of maximizing ex-ante utility in the three period model in which individuals are identical in period zero ($\pi$, resp. $1 - \pi$, of whom have income $y_h$, resp. $y_l$, in the first period). Since ex-ante expected utility amounts to a probability-weighted average, it can be thought of as a utilitarian objective. Second, Geanakoplos and Polemarchakis’ (1986) theorem allows for transfers of goods between consumers in the first period. Thus, as long as the change in the (weighted) sum of utilities is strictly positive after an infinitesimal perturbation of the asset holdings in equilibrium, we can make everyone better off by transferring goods in the initial period from those who gain to those who lose from the reallocation of assets.

**Definition 2.2.** An equilibrium $E(a) \equiv \{a^h(a), a^l(a), q^*(a)\}$ is constrained efficient if there is no $a'$ such that

$$\pi V^h(a') + (1 - \pi) V^l(a') > \pi V^h(a) + (1 - \pi) V^l(a)$$
where, for $i \in \{h, l\},$

$$V^i(a) = \max_{a' \geq 2} u\left(y_i - q^* a'\right) + \beta \left\{ \pi_i u\left(y_i + a'\right) + (1 - \pi_i) u\left(y_{-i} + a'\right) \right\}$$

Starting from the equilibrium $E(a) = \{a^{hs}(a), a^{is}(a), q^*(a)\}$ of the economy with borrowing constraint $a < 0$, suppose that the planner imposes a tighter borrowing constraint, $a' = a^{hs} + da$, so that the original equilibrium is no longer feasible. We first study some properties of the resulting equilibrium $E(a') = \{a^{bs}, a^{ls}, q'\}$. We start by studying the case where agent $l$ is not borrowing constrained in the initial equilibrium $E(a)$, so that (2.3) holds with equality before the tighter borrowing constraint is imposed.$^3$ This will be the case, in particular, if the initial borrowing constraint is the natural debt limit, $a = -y$, because of the Inada condition. More generally, agent $l$ is not borrowing constrained as long as $a$ is smaller than his optimal savings choice $a^{ls}(-y_l)$ in the economy with the natural borrowing constraint, $E(-y_l)$. Suppose therefore for simplicity that the initial borrowing constraint is $a = a^{ls}$ and the planner marginally tightens it to $a + da$.

**Lemma 2.3.** The new equilibrium price (after the tightening of the borrowing constraint) is strictly larger than the initial equilibrium price, i.e. $q' > q^*$.

**Proof.** Note first that we necessarily have $a' > a^{ls}$ because of the tighter borrowing constraint. Moreover, we still have $a^h < 0 < a^b$, because either the (new) borrowing constraint binds for agent $l$, so $a^b = a' < 0$, or it does not bind, but then the new equilibrium must be another equilibrium of the initial economy, for which we already proved that $a^b < 0 < a^h$. Market clearing therefore imposes $a^{b'} < a^{hs}$, because $a^b = \frac{1}{1 - \pi} a^{b} < \frac{1}{1 - \pi} a^{ls} = a^{hs}$.

Let us now differentiate $h$’s first-order condition (2.2), which holds with equality, around the original equilibrium $E(a)$, in order to obtain the effect of a marginal change in the price of the asset, $q$, on the savings choice of agent $h$. We get

$$\left\{ u'(y_h - q^* a^{hs}) - q^* a^{hs} u''(y_h - q^* a^{hs}) \right\} dq + \left\{ q^{2} a'' y_h - q^* a^{hs} u''(y_h - q^* a^{hs}) + \beta \pi_h u''(y_h + a^{hs}) + \beta (1 - \pi_h) u''(y_l + a^{hs}) \right\} da^h$$

Using $u' > 0$, $u'' < 0$, and $a^{hs} > 0$, we obtain that agent $h$’s asset holding is decreasing in the price of the asset, $q$, i.e.

$$\frac{da^h}{dq} = \gamma \quad \text{with} \quad \gamma < 0$$

$^3$The efficiency property of an equilibrium in which the borrowing constraint binds for agent $l$ will be studied in the next section.

$^4$Note that we wrote the differentiation around the initial equilibrium $E(a)$, but it is clear that the decreasing relationship between $q$ and the optimal savings choice $a^h$ holds as long as $a^b > 0$, which always holds.
Since in the new equilibrium $E(a')$ we have $a^{hl} < a^{hs}$, it is necessary that $q'$ be larger than $q^*$, in order to make sure that it is optimal for agent $h$ to save less than in the original equilibrium.

We now study the optimal savings choices of agents $h$ and $l$ in the new equilibrium $E(a')$. We already know that

$$a \leq a^l < a' \leq a^l' < 0 < a^{hl} < a^{hs}$$

Let us first look at the effect of a tightening of the borrowing constraint $a$ and the resulting increase in the asset price $q$ on the optimal saving choice of agent $l$. We have supposed that agent $l$ is not borrowing constrained in the original equilibrium, so that (2.3) holds with equality. Let us therefore differentiate (2.3) around the (unconstrained) initial equilibrium. We obtain

$$\left\{ u'(y_l - q^* a^l) - q^* a^l u''(y_l - q^* a^l) \right\} dq = \left\{ q^* u''(y_l - q^* a^l) + \beta \pi u''(y_l + a^l) + \beta(1 - \pi) u''(y_h + a^l) \right\} da$$

Since $a^l < 0$, the sign of the derivative of agent $l$’s asset holding with respect to the price of the asset is in general ambiguous:

$$\frac{da^l}{dq} = \delta \quad \text{with} \quad \delta \gtrless 0$$

Intuitively, since agent $l$ is borrowing in the first period, there are two opposite forces at play when the price of the asset increases, or equivalently when the interest rate decreases. Because it is cheaper to borrow, the substitution effect tends to decrease $a^l$, i.e. agent $l$ tends to borrow even more. But now the income effect goes in the opposite direction: agent $l$ can borrow more with the same amount of money, and hence tends to reduce his level of borrowing. With a log utility function (or with a CES utility function with risk aversion parameter $\sigma \leq 1$), it is easy to prove that the substitution effect dominates, and $\delta < 0$.

Let the planner imposes the tighter borrowing constraint $a'$, and consider the infinitesimal price change $d\tilde{q} > 0$ which induces agent $h$ to choose asset holdings $-\frac{1 - \pi}{\pi} a$, that is,$^5$

$$d\tilde{q} = \gamma^{-1} \left( -\frac{1 - \pi}{\pi} \right) da$$

There are two possible cases to consider.

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$^5$Recall that the initial level of $h$’s asset holdings was $a^{hs} = -\frac{1 - \pi}{\pi} a^l$, and that we denote $da \equiv a - a^{hs} > 0$. 
Case 1. If $d\bar{a} \equiv \delta d\bar{q} \leq d\bar{a}$, then the individual is borrowing constrained in the new equilibrium $E(a')$. The new equilibrium is \( \{a', a'^*, q^*\} = \{a', -\frac{(1-\pi)}{\pi}a', q^* + d\bar{q}\} \). It satisfies both agents’ first-order conditions ((2.3) is then an inequality), and market clearing.

Case 2. If $d\bar{a} \equiv \delta d\bar{q} > d\bar{a}$, then the previous allocation and price are no longer an equilibrium, because at this price agent l wants to borrow strictly less than $a'$, whereas agent h wants to save $-(1-\pi)\pi a'$, hence market clearing does not hold. This implies that $q$ has to increase more (making $h$ save strictly less) in order to restore market clearing. The new equilibrium \( \{a' h, a' l, q'\} \) will then be such that $a' l > a' > a' h$; hence, the new equilibrium is also an equilibrium of the original economy (with borrowing constraint $a$), which must therefore have had several equilibria in the first place.

We now prove that in both these cases, the new equilibrium dominates the original one from the utilitarian planner’s point of view; this will prove the constrained inefficiency of the original equilibrium $E(a)$.

We start with the second case, in which the economy “jumps” from $E(a) = \{a'h^*, a'^*, q^*\}$ to another one of its equilibria of the initial economy, $E(a') = \{a'b', a'^*, q'\}$, after a tightening of the borrowing constraint. We have $0 < a'b' < a'h^*$, $0 > a'^* > a'^*$, and $q' > q^*$. We prove in the Appendix that the jump to the new equilibrium makes agent l unambiguously better off, i.e. $V' l - V'^* > 0$, and makes agent $h$ unambiguously worse off, i.e. $V' h - V'^* < 0$. However, as we argued in the introduction to this section, in order to prove that the initial equilibrium is constrained inefficient, we need to prove that the sum of the changes in agents’ utilities is strictly positive, i.e. that the gain for agent l is larger than the loss for agent $h$. The proof of the following proposition is tedious, so we present it in the Appendix:

**Proposition 2.4.** The equilibrium $E(a')$ dominates the equilibrium $E(a)$ from the point of view of the utilitarian planner, if $a'^* > a'^*$. That is, 
\[
(1-\pi) \{V'^* - V'^*\} + \pi \{V'^* - V'^*\} > 0
\]

**Proof.** See the Appendix. \(\square\)

Now let us tackle the first case, in which the new equilibrium is $E(a') \equiv \{a', a'^*, q'\} = \{a', -\frac{(1-\pi)}{\pi}a', q^* + d\bar{q}\}$, where $d\bar{q} = \gamma^{-1} \left( -\frac{1-\pi}{\pi} \right) d\bar{a}$. Let us compute the differential change in agents’ utilities due to this perturbation of the equilibrium.
For agent \( l \), we have

\[
\left. \frac{dV^l}{da} \right|_{E(a)} = \left\{ -q^* u' (y_l - q^* a^{ls}) + \beta \pi_l u' (y_l + a^{ls}) + \beta (1 - \pi_l) u' (y_l + a^{ls}) \right\} \left. \frac{da^l}{da} \right|_{E(a)} \\
+ \left\{ -a^{ls} u' (y_l - q^* a^{ls}) \right\} \left. \frac{dq}{da} \right|_{E(a)}
\]

The first term of this expression is equal to zero, because \( a^{ls} \) is initially an optimum (recall that we assumed that in the initial equilibrium, agent \( l \) is not borrowing constrained, so that (2.3) holds with equality). This essentially follows from the envelope theorem: if prices did not change, forcing agent \( l \) to borrow a little less of the asset would hardly matter, since from the first-order condition the loss in consumption in the first period by agent \( l \) is almost exactly offset by the gain in the second period. But prices do change, and we obtain

\[
\left. \frac{dV^l}{da} \right|_{E(a)} = \left. \frac{dq}{da} \right|_{E(a)} = \gamma^{-1} \left( \frac{1 - \pi}{\pi} \right) a^{ls} u' (y_l - q^* a^{ls}) > 0
\]

(This expression is negative because both \( \gamma \) and \( a^{ls} \) are negative.) Thus agent \( l \) is made unambiguously better off by the tightening of the borrowing constraint and the resulting price change.

We can do the same calculation for agent \( h \) to find

\[
\left. \frac{dV^h}{da} \right|_{E(a)} = \left\{ -q^* u' (y_h - q^* a^{hs}) + \beta \pi_h u' (y_h + a^{hs}) + \beta (1 - \pi_h) u' (y_h + a^{hs}) \right\} \left. \frac{da^h}{da} \right|_{E(a)} \\
+ \left\{ -a^{hs} u' (y_h - q^* a^{hs}) \right\} \left. \frac{dq}{da} \right|_{E(a)}
\]

\[
= \left. \frac{dq}{da} \right|_{E(a)} = \gamma^{-1} \left( \frac{1 - \pi}{\pi} \right) a^{hs} u' (y_h - q^* a^{hs}) < 0
\]

Thus agent \( h \) is made unambiguously worse off by the tightening of the borrowing constraint and the resulting price change.

Now, we find the differential change in the utilitarian social welfare function by

\[
(1 - \pi) \left. \frac{dV^l}{da} \right|_{E(a)} + \pi \left. \frac{dV^h}{da} \right|_{E(a)} = (1 - \pi) \gamma^{-1} \left( \frac{1 - \pi}{\pi} \right) a^{ls} u' (y_l - q^* a^{ls}) \\
+ \pi \gamma^{-1} \left( \frac{1 - \pi}{\pi} \right) a^{hs} u' (y_h - q^* a^{hs})
\]

\[
= \pi \gamma^{-1} \left( \frac{1 - \pi}{\pi} \right) a^{hs} \left\{ u' (y_h - q^* a^{hs}) - u' (y_l - q^* a^{ls}) \right\}
\]
We prove in the Appendix (in the proof of Proposition 2.4), that 

\[ y_l - q^*a^* < y_h - q^*a^h. \]

Since \( \gamma < 0 \), we finally obtain that

\[
(1 - \pi) \left. \frac{dV^l}{da} \right|_{E(a)} + \pi \left. \frac{dV^h}{da} \right|_{E(a)} > 0
\]

Therefore, we have proved the following proposition:

**Proposition 2.5.** The initial equilibrium \( E(a) \) is constrained inefficient.

What is the intuition underlying this result? As is usual in incomplete markets models, prices are not set optimally in the equilibrium, due to the existence of pecuniary externalities. Marginally tightening the borrowing constraint has no first-order effect on welfare through distortions of asset allocations, because we start from an equilibrium allocation. But it decreases the interest rate (or increases the price of the asset, \( q \)). This helps the low income agents, who are net borrowers, and hurts the high income agents, who are net savers. The average effect is positive, essentially because the marginal utility of the high income agents is lower than that of the low income agents, due to the concavity of the utility function.

Note that we have only discussed the case where agent \( l \) is unconstrained in the initial equilibrium. If he is borrowing constrained at \( a \), then (2.3) is an inequality, and the effect of the tightening of the borrowing constraint on the change in utility due to the change in assets (keeping prices constants) is negative and no longer of second-order; hence if this effect is large enough, i.e. if the constrained agent \( l \) is far from his optimal choice of assets, our perturbation will not increase the total (utilitarian) utility. We are going to characterize the constrained optimum in the next section, i.e. the optimal choice of \( \bar{a} \) by the government. We saw that agent \( l \) has to be borrowing constrained in this “second-best” allocation. Therefore, all equilibria (constraining \( l \) or not) in which \( a < \bar{a} \) are constrained inefficient and the optimal policy is to tighten the borrowing constraint; all equilibria in which \( a > \bar{a} \) are constrained inefficient and the optimal policy is to loosen the borrowing constraint.

### 2.4 The Constrained Optimum

The planner maximizes the utilitarian social welfare function subject to market clearing and agent \( h \)'s first-order condition (which must hold with equality). The latter constraint ensures that the price of the asset must adjust so that agent \( h \) is indeed optimizing by choosing to hold \( a^h = -(1 - \pi)a/\pi \).

Thus the constrained optimum \((\bar{q}, \bar{a})\) is the solution to the problem:
\[ \max_{q,a \leq 0} \pi \{ u(y_h - qa^h) + \beta \pi_h u(y_h + a^h) + \beta(1 - \pi_h)u(y_l + a^h) \} \]
\[ + (1 - \pi) \{ u(y_l - qa) + \beta \pi_l u(y_l + a) + \beta(1 - \pi_l)u(y_l + a) \} \]
\[ \text{s.t. } \pi a^h + (1 - \pi)a = 0 \]
\[ \text{and (2.2)} \]

So the optimal value of \( a \) must be chosen to satisfy

\[ \pi \{ qu'(y_h - qa^h) - \beta \pi_h u'(y_h + a^h) - \beta(1 - \pi_h)u'(y_l + a^h) \} \]
\[ + (1 - \pi) \{ qu'(y_l - qa) - \beta \pi_l u'(y_l + a) - \beta(1 - \pi_l)u'(y_h + a) \} \]
\[ = - \{ \pi a^h u'(y_h - qa^h) + (1 - \pi)qa'u'(y_l - qa) \} \frac{dq}{da} \]
\[ = (1 - \pi) a \left\{ u'(y_h - qa^h) - u'(y_l - qa) \right\} \frac{dq}{da} \]

where \( dq/da = -\gamma^{-1} \) is given by differentiating (2.2). Note that by the envelope theorem, the first term on the left hand side of the latter expression is zero by the envelope theorem. The second term in non-zero, because individual \( l \) must be borrowing constrained at the optimum. Intuitively, the change in asset allocation no longer has a second-order effect on agent \( l \)'s welfare. At the optimum, the adverse effect due to the asset reallocation must be equal to the positive effect through the increase in the asset price.

We therefore obtain the two equations characterizing the constrained optimum:

\[ \tilde{q}u'(y_l - \tilde{q}a) - \beta \pi_l u'(y_l + \tilde{a}) - \beta(1 - \pi_l)u'(y_h + \tilde{a}) \]
\[ = \tilde{a} \left\{ u'(y_l + \tilde{q} \left( \frac{1 - \pi}{\pi} \right) \tilde{a}) - u'(y_l - \tilde{q}a) \right\} \]
\[ \times \frac{\tilde{q}^2 u''(y_h + \tilde{q} \left( \frac{1 - \pi}{\pi} \right) \tilde{a}) + \beta \pi_h u''(y_h - \left( \frac{1 - \pi}{\pi} \right) \tilde{a}) + \beta(1 - \pi_h)u''(y_l - \left( \frac{1 - \pi}{\pi} \right) \tilde{a})}{u'(y_h + \tilde{q} \left( \frac{1 - \pi}{\pi} \right) \tilde{a}) + \tilde{q} \left( \frac{1 - \pi}{\pi} \right) \tilde{a} u''(y_h + \tilde{q} \left( \frac{1 - \pi}{\pi} \right) \tilde{a})} \]
\[ \tilde{q}u'(y_h + \tilde{q} \left( \frac{1 - \pi}{\pi} \right) \tilde{a}) = \beta \pi_h u'(y_h - \left( \frac{1 - \pi}{\pi} \right) \tilde{a}) + \beta(1 - \pi_h)u'(y_l - \left( \frac{1 - \pi}{\pi} \right) \tilde{a}) \]

The solution to these two equations gives the value of the two unknowns, \( \tilde{q} \) and \( \tilde{a} \). In general, it is hard to derive properties of the constrained optimum from these equations. Therefore, in the sequel, we focus on numerical simulations.
2.5 A Numerical Illustration

We first simulate the two-period model for $y_l = 1$ and $y_h = 2$. The initial distribution of endowments is symmetric, i.e., half of the agents start with each possible endowment level. Since we have already proved theoretically that an increase in the debt limit induces an increase in ex-ante welfare, the question we want to address is how the strength of this result depends on the structure of the model. Namely, here we are looking at how it is affected by changes in the persistence of the shocks. We consider values of $\pi_h = \pi_l \in [0.5, 1]$, i.e., several persistence parameters ranging from i.i.d. shocks to permanent shocks. The welfare impact of a 1% increase in the debt limit (starting from the natural debt limit $a = -y_l$) is shown in figure 1 below.\footnote{We should emphasize that the effects on welfare are rather small, since we were interested in simulating a marginal perturbation of the original economy. One may be worried about computational imprecisions being specifically relevant to such a small size result, but the order of magnitude of utility gains ($10^{-4}$) is much greater than that of the maximization problems considered ($10^{-10}$).}

![Figure 1: 2 period model - effect of persistence](image)

The constrained inefficiency result is due to the impact of the debt limit on bond prices, which affect agents in a way that is similar to insurance payments – they affect positively the initially poor agents. This observation suggests that the impact of the tightening of the borrowing constraint is related to the number of people using the bond market as a limited insurance mechanism in the economy with no intervention.
As endowment shocks become more persistent, there are less gains from smoothing consumption across periods. In the limit, as shocks are fully permanent, both types of agents have zero bonds since their marginal utility is constant. Because the effect of policies like the one we consider is due to their impact on asset prices, their efficacy is hindered if the bond market is used less.

This raises an interesting point. The constrained inefficiency result depends on the use of the already existing (incomplete) markets for limited insurance by agents. In the case of fully permanent endowment levels, there is very strong income uncertainty from an ex-ante perspective (that the government would want to reduce in a first best situation) but nothing can be done since the bond market does not help agents self-insure in equilibrium.

### 3 The Infinite-Horizon Economy

In the theoretical two-period model, we saw that the incomplete markets equilibrium allocation is constrained inefficient. This is so because an increase in the borrowing constraint has a direct effect on the allocation of agents through a reduction in the amount of debt held by the (initially) poor agents and a reduction in the amount of savings held by the (initially) rich agents, but also because of an indirect effect through bond prices. More specifically, the rate of return on bonds will go down with a higher borrowing constraint since the demand side of the market shrinks, and equilibrium then requires lower savings in the economy; this can only be achieved through less attractive rates of return. It is important to note here that if we look at a small perturbation of the borrowing constraint around the equilibrium with natural debt limit, the former effect has no first order impact since both agents were optimizing with respect to the debt/asset level, and hence only the effect through prices matters. This is basically the insight present in Geanakoplos and Polemarchakis (1986).

A decrease in the interest rate has positive effects on the (initially) poor agents, since they were the ones with a negative position on the bonds, and it affects negatively the rich agents who had a positive position on the bonds. Therefore the change in prices acts in the optimal direction of redistribution in this economy. This is the same rationale that goes through in the model of Davila, Hong, Krusell and Rios-Rull (2010) in a similar model with capital.

Following their steps, the next question to ask is whether the two-period model insights generalize to the infinite period model. In this case, the effects of the policy change will have much more complex effects. A uniform increase in the borrowing level should, at first sight, increase the bond prices for the same reasons presented in the two-period example.

We first answer this question numerically. We calculate the steady-state of the Huggett economy with a natural debt limit, and then consider marginal increases of the...
borrowing constraint.

The model is the extension of the two-period model to an infinite horizon. There is an infinite number of periods \( t = 1, 2, ... \) and a continuum of agents who receive random endowments \( y_t \in \{y_l, y_h\} \) in each period. As in the two-period model, shocks follow a first order Markov process, with a transition matrix given by \( P(y_{t+1} = y_h | y_t = y_h) = \pi_h \) and \( P(y_{t+1} = y_l | y_t = y_l) = \pi_l \). We assume that the initial distribution of shocks in the population is the ergodic distribution \(^7\) given by the transition (it will not matter for steady state calculation).

Given a borrowing constraint level \( a \), an agent’s problem given bond prices \( \{q_t\}_{t=1}^{\infty} \) is given by

\[
\max_{\{c_t, a_t\}} \mathbb{E}_0 [\sum_{t=1}^{\infty} \beta^{t-1} u(c_t)] \\
\text{s.t. } \begin{cases} c_t + q_t a_{t+1} \leq y_t + a_{t-1} \\ a_{t+1} \geq a \\ a_0 = 0 \end{cases}, \text{ for all } t \geq 1.
\]

In the setup we are analyzing, equilibrium prices will have a recursive structure. More specifically, since we do not have aggregate shocks, prices in the steady-state will be just a fixed number. In this case we can write the optimization problem of an agent, given savings \( a \geq a_0 \) and current endowment \( y_t \), in a recursive way. Let us denote as \( V^q(a) \) the value function of an agent with savings \( a \) when the (constant) bond price is given by \( q \geq 0 \). It satisfies:

\[
V^q(a, y) = \max_{c, a' \geq a} u(c) + \beta \mathbb{E} [V^q(a', y') | y] \\
\text{s.t. } c + a \leq qa' + y
\]

We denote as \( a'_q(a, y) \) as the policy function that solves this problem.

Define \( \Delta (\{y_l, y_h\} \times [a, \infty)) \equiv \Delta \) as the set of joint distribution over endowment and savings levels. Given a policy function, we can define \( T_q : \Delta \to \Delta \) as the updating rule for the distribution of wealth in the economy by

\[
T_q \chi (\{y_i\} \times B) = \pi_i \chi \left( \{y_i\} \times \{a : a'_q(a, y_i) \in B\} \right) + (1 - \pi_j) \chi \left( \{y_j\} \times \{a : a'_q(a, y_j) \in B\} \right),
\]

for any \( i \neq j \) and \( B \in B(\mathbb{R}) \).

A **steady-state equilibrium** is a bond price \( q^* \geq 0 \) and a distribution \( \chi^* \in \Delta (\{y_l, y_h\} \times [a, \infty)) \) such that

\[
T_{q^*} \chi^* = \chi^*; \\
\int a d\chi^* = 0.
\]

\(^7\)This means that \( \pi = \frac{1 - \pi_l}{2 - \pi_h - \pi_l} \) and \( 1 - \pi = \frac{1 - \pi_h}{2 - \pi_h - \pi_l} \), where \( \pi \) is the initial share of high endowment agents.
3.1 Parameters and Results

Our focus is on the qualitative rather than quantitative results, and this is reflected in the parametrization of the model we calculated. Our main goal is to check whether the insights obtained in the two-period model continue to hold in an infinite horizon model; we do not try to assess the magnitude of the effects. In our calculations, we use the utility function \( u(c) = \log c \). The transition function is given by \( \pi_h = \pi_l = 0.6 \). The endowment levels are \( y_l = 1 \) and \( y_h = 2 \). The discount factor is \( \beta = 0.9 \). We use a grid for asset levels with 800 values, even though we tried other numbers and our results were not affected.

We first calculate the equilibrium amount with the so-called “natural debt limit”, which is given by

\[
a_n = -\frac{y_l}{1 - q},
\]

It is the amount of debt that anyone can pay for sure if they allocate all their future income for debt payment.

We computed the steady state under the natural debt limit, and imposed marginal changes in the debt limit above this level. We calculated the 10th percentile of the steady state savings distribution, which is a negative number, and calculated the equilibrium for a grid of borrowing requirements between this amount and the natural level.

One can check in Figure 1 that, as in the two-period model, the bond price increases with the borrowing limit imposed on agents. The rationale for this is simple: with a tighter debt limit, people on the lower tail of the asset distribution are pushed to have less debt, so in an equilibrium the wealthy agents need to be accumulating less savings; this is achieved through a higher bond price or lower rate of return. In figure 2 we plot the wealth distribution in this economy for three different debt limits: the natural one, an intermediary one, and the highest point in our grid (which is the 10th percentile of the original equilibrium’s distribution).

The result, as expected, is an upward shift in the asset level for all agents in this economy. By definition, no one will borrow more than the debt limit; this already influences the left tail of the distribution. Moreover, this effect also occurs for levels near the debt limit. Indeed, agents do not want to get too close to the debt limit, since a negative sequence of shocks can take them to the highest possible debt level they can have, and it is then impossible for them to buffer any further negative shocks received. One interesting fact is that the amount of debtors does not change a lot in our calculations; we always have approximately 60% of agents holding negative assets in the steady state.

The relevant question for our purposes is the welfare effect of the tightening of the borrowing constraint. If we consider welfare from an utilitarian perspective, there is a clear increase from imposing more severe debt limits in this economy, as one can see
from figure 3. We plot average (total) utility, and average utilities conditional on each current endowment shock. Here the persistence of shocks is very low, which means that the current shock has a small effect on utility level (it is very close to a temporary shock).

We are only able to compare steady-state levels, rather than transitions, due to the technical difficulties of computing the transition path. But for our specific question, this is very problematic. Indeed, the increase in utility calculated across steady-states reflects very much the fact that average wealth has increased in the new steady state. A higher debt limit implies that in the new steady state more people will have higher levels of assets, and therefore will be better off. We expect that transitions should matter a lot. Agents that were very close to the old debt limit will have to endure several periods of paying off debt to increase their asset position to comfortable levels, given the new requirement. This is going to have large welfare costs and we are unable to see this in our calculation. Our metric would be reasonable in the case of an utilitarian social planner who only cares about long run effects of the policy. It is clear that in the long-run society will be better off because of the induced effort to accumulate assets generated by the policy.

An alternative is to compare people with identical asset levels. This means that we are considering only the price effects on agents with the same given savings level. We can do this by comparing the value functions for different steady state prices. They are
plotted in figure 4. We show the value function for an agent with current low endowment. The values are very close to those of high endowment agents – given the low persistence assumption – and the changes due to the tightening of the debt limit are the same. This calculation confirms that the effects on average utility are completely driven by changes in the steady state wealth distribution because utility levels for a given asset position are unchanged. The order of magnitude of changes in expected utility is larger than that of changes in value functions.

The changes in the value function actually move in the opposite direction. This means that an agent with the same asset position is slightly worse off with the increased debt limit, since savings are more expensive and he has smaller freedom to buffer possible future negative shocks. This is a more important comparison since the constrained inefficiency results in the two-period model are driven by the effects of price changes, but price changes affect negatively the utility of all agents in this economy. It seems that the difference is that the initially poor agents were better off because they were net sellers of bonds in the economy, and so enjoy higher gains from an increase in bond price. In a fully dynamic model, higher bond prices might be a negative change for everyone since they consider the possibility that in the future they might be bond holders.

Another relevant observation is that the results obtained in the two-period model depend crucially in the small perturbation assumption. It is necessary that only price changes have first order effects on agents. It could be the case that we are looking
at policy changes that are too big, and therefore that the impact of a stricter debt requirement is also first-order and dominates the effect of price changes on agents. This does not seem to be the case since the results are monotone in the debt requirement level and we have considered very small changes in the debt requirement.

4 Conclusion

The objective of this paper was to understand whether the insights learned in Davila, Hong, Krusell and Rios-Rull (2010) regarding the constrained inefficiency property of general equilibrium models with incomplete markets are specific to the under-accumulation of capital present in their setting. We find that the same result holds true in a Huggett model, where there is no capital accumulation, but there is a market for riskless bond. Since there is no capital choice, we have looked at a similar intervention in nature, which is the increase in debt limit imposed in the economy. The rationale for this is that this will push poor agents to borrow less, and this will have an effect in the rate of return on the market. For very small interventions, the envelope conditions allow us to consider only the effect of prices on welfare of the agents. This effect works as follows. The tighter credit limit lowers the interest rate in the economy, which affects negatively the initially rich agents – the lenders – and affects positively the initially poor

Figure 4: Average utility
agents - the borrowers. Since the effects are positive for agents with higher marginal utility, this leads to a gain in ex-ante utility.

There are two possible motivations for considering ex-ante utility. First, one can reinterpret our model as a three period model in which in the first period agents are identical. Second, using a utilitarian social welfare function may be debatable for redistributive concerns, but it may still offer important insights.

We find that the constrained inefficiency result would fail in the case of fully permanent shocks. The problem is that in this case no one would use bonds to smooth consumption in the economy since their marginal utility is already constant. In this case the effects of the policy through prices do not work since all agents are indifferent (in the margin) about the interest rate since they have zero net wealth in bonds. This situation is the one in which the insurance problem is most severe; nonetheless there is no possible way for the social planner to increase welfare. The problem is that the effect of the policy studied here relies crucially on the use of the bond market.

The relevant question is which of these results remain true in a fully dynamic model with endowment shocks, as in Huggett (1993). We have numerically solved for the steady state in the infinite period version of the model with a natural debt limit. Then we have considered small changes in the debt requirement and checked how it affected welfare of different agents. The result is that the aggregate social welfare (the sum of utilities) increases, as well as aggregate welfare conditional on each endowment types.
These results are driven by changes in the steady state wealth distribution, which is shifted toward higher asset positions.

We also found that the change in prices affects negatively all agents, even though the effects are small. This suggests that an analysis including transition dynamics following the policy might lead to different conclusions. Even though we did not consider these effects, they seem to be very important for our question and can potentially change the welfare impacts. Our opinion is that there is no reason to believe that the insights from the two-period model should survive in an infinite period model with the same strength.

4.1 Extensions

It is very important to look at our inefficiency result in a broader perspective. We have focused exclusively on one possible policy and assumed one specific market structure. An interesting exercise would be to take a mechanism design approach, by modelling the more basic informational, physical and contractual frictions present in the economy and then considering optimal allocation in such settings.

One could for instance try to look at the model proposed by Cole and Kocherlakota (2001), in which there are unobservable endowment shocks as in the model considered here, and individuals have access to an unobservable storage technology. The presence of unobservable storage hinders the existence of insurance in this economy, since it allows people to get higher gains from lying about their shocks. One possible thing to do is to consider such a storage technology as an exogenous risk free bond market, in which the debt limit is $a = 0$. In any other case (with $a$ not necessarily zero), it is a valid question to ask how the possibilities of insurance in society increase with a different debt limit.

If the debt limit were negative, which means that people have access to limited borrowing, welfare can only get worse since this increases the possibility of deviations by agents in society from the mechanism designer (an insurance company or the government). In this model, it is always a better option for the government to hold all savings in society.

The other direction would also have no effect. If we considered an increase in the debt limit in the (unobservable) bond market, the possibilities of insurance for the economy remain the same. This is because the binding constraint in the model is for wealthy agents to lie about their endowment in order to receive more transfers, and then saving part of their gains from such a strategy. These agents would not be affected by a higher debt limit since they are net lenders in the bond market. From a private information perspective, constraining the low endowment agents to have access to markets has no effect on the optimal allocation, since they are not the ones who have incentives to lie.
REFERENCES

References


A Appendix

A.1 Proof of Lemma 2.1

Note first that if \( \pi_l = 1 \), i.e. having low income is an absorbing state, then we have, from (2.3)
\[
q^* u' \left( y_l - q^* a^{ls} \right) \geq \beta u' \left( y_l + a^{ls} \right)
\]
But since \( a^{ls} < 0 \) and \( q^* > 0 \), we have \( u' \left( y_l - q^* a^{ls} \right) < u' \left( y_l + a^{ls} \right) \). Thus it must be the case that \( q^* > \beta \).

Now suppose \( \pi_l < 1 \). From (2.3), we have, using the fact that \( u' \) is convex,
\[
\frac{q}{\beta} \geq \pi_l \frac{u'(y_l + a^{ls})}{u'(y_l - qa^{ls})} + (1 - \pi_l) \frac{u'(y_h + a^{ls})}{u'(y_l - qa^{ls})} \geq \frac{u'(\pi_l y_l + (1 - \pi_l) y_h + a^{ls})}{u'(y_l - qa^{ls})}
\]
So we want to prove that there exists \( \eta > 0 \) such that for \( \pi_l > 1 - \eta \), then
\[
\pi_l y_l + (1 - \pi_l) y_h + a^{ls} \leq y_l - qa^{ls}
\]
Since \( a^{ls} < 0 \) and \( q^* > 0 \), it is sufficient to prove that for \( \pi_l \) close enough to 1, \( \pi_l y_l + (1 - \pi_l) y_h + a^{ls} \leq y_l \), i.e. \( (1 - \pi_l)(y_h - y_l) \leq |a^{ls}| \). But note that, for \( a < 0 \), there exists \( \eta' > 0 \) such that all equilibria of the economy have \( a^{ls} < -\eta' < 0 \). Indeed, we otherwise would have a sequence of equilibria \( \{E_n(a)\}_{n \geq 0} \) such that \( a_n^{ls} \to_{n \to \infty} 0 \). By market clearing, we must have \( a_n^{hs} \to_{n \to \infty} 0 \). Since the equality (2.2) and the inequality (2.3) hold for all those equilibria \( \{a_n^{hs}, a_n^{ls}, q_n^*\} \), by continuity they also hold for \( \lim a_n^{hs}, \lim a_n^{ls}, \lim q_n^* \) (note that, as we will see later, \( q_n^* \) is an increasing sequence, hence it has a limit, possibly \( +\infty \)). But this can not be true, as we proved above.

Now, there exists \( \eta > 0 \) such that for \( \pi_l > 1 - \eta \), we have \( (1 - \pi_l)(y_h - y_l) \leq \eta' < |a^{ls}| \). We conclude that if the low shock \( y_h \) is sufficiently persistent, then the equilibrium price necessarily satisfies \( q^* > \beta \).

A.2 Proof of Proposition 2.4

We have
\[
V^{l'} - V^{ls} = u \left( y_l - q' a^{l'} \right) + \beta \left\{ \pi_l u \left( y_l + a^{l'} \right) + (1 - \pi_l) u \left( y_h + a^{l'} \right) \right\} \\
- u \left( y_l - q^* a^{ls} \right) - \beta \left\{ \pi_l u \left( y_l + a^{ls} \right) + (1 - \pi_l) u \left( y_h + a^{ls} \right) \right\} \\
= \left\{ u \left( y_l - q' a^{l'} \right) - u \left( y_l - q^* a^{ls} \right) \right\} + \beta \pi_l \left\{ u \left( y_l + a^{l'} \right) - u \left( y_l + a^{ls} \right) \right\} \\
+ \beta (1 - \pi_l) \left\{ u \left( y_h + a^{l'} \right) - u \left( y_h + a^{ls} \right) \right\}
\]
Let us look at the first line on the right hand side. Since $h$ is made unambiguously worse off by the jump to the new equilibrium, induced by the tightening of the borrowing constraint. Similarly, we have, using $a^{h'} < a^{hs}$ and $q' a^{h'} > q' a^{hs}$,

$$\begin{align*}
V^{h'} - V^{hs} &\equiv u (y_l - q'a^{h'}) + \beta \left\{ \pi_h u (y_h + a^{h'}) + (1 - \pi_h) u (y_l + a^{hs}) \right\} \\
&\quad - u (y_l - q'a^{hs}) - \beta \left\{ \pi_h u (y_h + a^{hs}) + (1 - \pi_h) u (y_l + a^{hs}) \right\} \\
&= \left\{ u (y_h - q'a^{h'}) - u (y_h - q'a^{hs}) \right\} + \beta \pi_h \left\{ u (y_h + a^{h'}) - u (y_h + a^{hs}) \right\} \\
&\quad + (1 - \pi_h) \left\{ u (y_l + a^{h'}) - u (y_l + a^{hs}) \right\} \\
&< 0
\end{align*}$$

i.e. agent $h$ is made unambiguously worse off by the jump to the new equilibrium.

Let us now prove that

$$(1 - \pi) \left\{ V^{h'} - V^{ls} \right\} + \pi \left\{ V^{hs} - V^{hs} \right\} > 0$$

We have

$$(1 - \pi) \left\{ V^{h'} - V^{ls} \right\} + \pi \left\{ V^{hs} - V^{hs} \right\}$$

$$= (1 - \pi) \left\{ u (y_l - q'a^{h'}) - u (y_l - q'a^{ls}) \right\} + \pi \left\{ u (y_h - q'a^{h'}) - u (y_h - q'a^{hs}) \right\}$$

$$+ (1 - \pi) \beta \pi_l \left\{ u (y_l + a^{h'}) - u (y_l + a^{ls}) \right\} + \pi \beta (1 - \pi_h) \left\{ u (y_l + a^{h'}) - u (y_l + a^{hs}) \right\}$$

$$+ (1 - \pi) \beta (1 - \pi_l) \left\{ u (y_h + a^{h'}) - u (y_h + a^{ls}) \right\} + \pi \beta \pi_h \left\{ u (y_h + a^{h'}) - u (y_h + a^{hs}) \right\}$$

(A.1)

Let us look at the first line on the right hand side. Since $u$ is concave and $q' a^{h'} < q' a^{ls}$, we have

$$\begin{align*}
(1 - \pi) \left\{ u (y_l - q'a^{h'}) - u (y_l - q'a^{ls}) \right\} &\geq (1 - \pi) \left( q' a^{ls} - q' a^{h'} \right) u' (y_l - q'a^{ls}) \\
&= \pi (q' a^{h'} - q' a^{hs}) u' (y_l - q'a^{ls})
\end{align*}$$
And, since $q^*a^{h'} > q^*a^{h*}$,

$$
\pi \left\{ u \left( y_h - q^*a^{h'} \right) - u \left( y_h - q^*a^{h*} \right) \right\} = -\pi \left\{ u \left( y_h - q^*a^{h'} \right) - u \left( y_h - q^*a^{h*} \right) \right\} \\
\geq -\pi \left( q^*a^{h'} - q^*a^{h*} \right) u' \left( y_h - q^*a^{h*} \right)
$$

Hence

$$(1 - \pi) \left\{ u \left( y_l - q^*a^{l'} \right) - u \left( y_l - q^*a^{l*} \right) \right\} + \pi \left\{ u \left( y_h - q^*a^{h'} \right) - u \left( y_h - q^*a^{h*} \right) \right\} \\
\geq \pi \left( q^*a^{h'} - q^*a^{h*} \right) \left\{ u' \left( y_l - q^*a^{l*} \right) - u' \left( y_h - q^*a^{h*} \right) \right\}
$$

But $y_l - q^*a^{l*} < y_h - q^*a^{h*}$. Indeed, from the first-order condition (2.3), we have

$$q^*u' \left( y_l - q^*a^{l*} \right) \geq \beta\pi_l u' \left( y_l + a^{l*} \right) + (1 - \pi_l)u' \left( y_h + a^{l*} \right)
$$

But since $a^{l*} < 0 < a^{h*}$, we have $u' \left( y_l + a^{l*} \right) > u' \left( y_l + a^{l*} \right)$ and $u' \left( y_l + a^{l*} \right) > u' \left( y_h + a^{h*} \right)$. Moreover, we have assumed that $\pi_l > 1 - \pi_h$. Hence

$$\beta\pi_l u' \left( y_l + a^{l*} \right) + (1 - \pi_l)u' \left( y_h + a^{l*} \right) \geq \beta(1 - \pi_h)u' \left( y_l + a^{l*} \right) + \beta\pi_h u' \left( y_h + a^{h*} \right) \\
\geq \beta(1 - \pi_h)u' \left( y_l + a^{h*} \right) + \beta\pi_h u' \left( y_h + a^{h*} \right) \\
= q^*u' \left( y_h - q^*a^{h*} \right)
$$

This proves that $u' \left( y_l - q^*a^{l*} \right) > u' \left( y_h - q^*a^{h*} \right)$, hence $y_l - q^*a^{l*} < y_h - q^*a^{h*}$, and therefore

$$(1 - \pi) \left\{ u \left( y_l - q^*a^{l'} \right) - u \left( y_l - q^*a^{l*} \right) \right\} + \pi \left\{ u \left( y_h - q^*a^{h'} \right) - u \left( y_h - q^*a^{h*} \right) \right\} > 0
$$

Now look at the second line on the right-hand side of the expression (A.1). We have, using $a'' > a^*$ and the fact that $u$ is concave,

$$(1 - \pi)\pi_l \left\{ u \left( y_l + a^{h'} \right) - u \left( y_l + a^{h*} \right) \right\} \geq (1 - \pi)\pi_l \left( a'' - a^* \right) u' \left( y_l + a^{l'} \right) \\
= \pi_l \left( a^{h*} - a^{h'} \right) u' \left( y_l + a'' \right)
$$

Similarly, using $a^{h'} < a^{h*}$,

$$\pi(1 - \pi_h) \left\{ u \left( y_l + a^{h'} \right) - u \left( y_l + a^{h*} \right) \right\} = -\pi(1 - \pi_h) \left\{ u \left( y_l + a^{h*} \right) - u \left( y_l + a^{h'} \right) \right\} \\
\geq -\pi(1 - \pi_h) \left( a^{h*} - a^{h'} \right) u' \left( y_l + a^{h*} \right)
$$

Doing the same steps on the third line of (A.1) gives

$$(1 - \pi)(1 - \pi_l) \left\{ u \left( y_h + a^{l'} \right) - u \left( y_h + a^{l*} \right) \right\} \geq (1 - \pi)(1 - \pi_l) \left( a'' - a^* \right) u' \left( y_h + a^{l'} \right) \\
= \pi(1 - \pi_l) \left( a^{h*} - a^{l'} \right) u' \left( y_h + a'' \right)$$
And finally,
\[ \pi \pi_h \{ u(y_h + a^{h'}) - u(y_h + a^{h*}) \} = -\pi \pi_h \{ u(y_h + a^{h'}) - u(y_h + a^{h*}) \} \]
\[ \geq -\pi \pi_h \{ a^{h*} - a^{h'} \} u'(y_h + a^{h'}) \]

Therefore, we obtain
\[
(1 - \pi) \beta \pi_l \{ u(y_l + a^{l'}) - u(y_l + a^{l'}) \} + \pi \beta (1 - \pi_l) \{ u(y_l + a^{l'}) - u(y_l + a^{l'}) \} \\
+ (1 - \pi) \beta (1 - \pi_l) \{ u(y_h + a^{h'}) - u(y_h + a^{h'}) \} + \pi \beta \pi_h \{ u(y_h + a^{h'}) - u(y_h + a^{h'}) \} \\
\geq \beta \pi (a^{h*} - a^{h'}) \{ \pi_l u' (y_l + a^{l'}) - (1 - \pi_l) u' (y_l + a^{l'}) \} \\
+ \beta \pi (a^{h*} - a^{h'}) \{ (1 - \pi_l) u' (y_h + a^{h'}) - \pi_h u' (y_h + a^{h'}) \} \\
= \beta \pi (a^{h*} - a^{h'}) \left[ \{ \pi_l u' (y_l + a^{l'}) - (1 - \pi_l) u' (y_l + a^{l'}) \} \right. \\
\left. - \{ (1 - \pi_h) u' (y_l + a^{l'}) + \pi_h u' (y_h + a^{h'}) \} \right] \\
\]

Since \( \pi_l \geq 1 - \pi_h \) and \( u' (y_l + a^{l'}) > u' (y_h + a^{h'}) \), we have
\[ \pi_l u' (y_l + a^{l'}) - (1 - \pi_l) u' (y_l + a^{l'}) \geq (1 - \pi_h) u' (y_l + a^{l'}) + \pi_h u' (y_h + a^{h'}) \]

And thus the previous expression is larger than
\[
\beta \pi (a^{h*} - a^{h'}) \left[ \{ (1 - \pi_h) u' (y_l + a^{l'}) + \pi_h u' (y_h + a^{h'}) \} \right. \\
\left. - \{ (1 - \pi_h) u' (y_l + a^{l'}) + \pi_h u' (y_h + a^{h'}) \} \right] \\
= \beta \pi (a^{h*} - a^{h'}) \left[ (1 - \pi_h) \{ u' (y_l + a^{l'}) - u' (y_l + a^{l'}) \} + \pi_h \{ u' (y_h + a^{h'}) - u' (y_h + a^{h'}) \} \right] \\
\]

But since \( a^{h*} > a^{h'} \), \( u' (y_l + a^{l'}) > u' (y_l + a^{l'}) \) and \( u' (y_h + a^{h'}) > u' (y_h + a^{h'}) \), this expression is positive. We can therefore conclude:
\[ (1 - \pi) \{ V^{l'} - V^{l*} \} + \pi \{ V^{h'} - V^{h*} \} > 0 \]